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The superstring Hagedorn temperature in a pp-wave background

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Abstract: The thermodynamics of type-IIB superstring theory in the maximally supersymmetric plane wave background is studied. We compute the thermodynamic partition function for non-interacting strings exactly and the result differs slightly from previous computations. We clarify some of the issues related to the Hagedorn temperature in the limits of small and large constant RR 5-form. We study the thermodynamic behavior of strings in the case of $AdS_3 \times S^3 \times T^4$ geometries in the presence of NS-NS and RR 3-form backgrounds. We also comment on the relationship of string thermodynamics and the thermodynamic behavior of the sector of Yang-Mills theory which is the holographic dual of the string theory.

Keywords: String Duality, Penrose limit and pp-wave background.

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Contents

1. Introduction

Understanding the finite temperature states of string theory is essential for many of its potential applications, particularly the study of black holes and early universe cosmology. One of the fascinating features exhibited by string theories is the exponential growth of their densities of states with energy [1]. For their thermodynamics this leads to either a limiting, Hagedorn temperature beyond which an ensemble of strings cannot be heated or perhaps a phase transition to a state which is better described by degrees of freedom other than strings [2].

The existence of a Hagedorn temperature is well-established for all consistent non-interacting string theories on Minkowski space. Recently, it has been noted that the non-interacting type-IIB superstring can be solved explicitly on a maximally supersymmetric plane-wave background [3, 4]. This gives a background other than Minkowski space where some of the ideas of string theory can be tested. Indeed, there are now several discussions of the Hagedorn temperature on the plane-wave background where it has been shown to be a non-trivial function of the dimensional parameter of the background and the string scale [5]–[9].

Another fascinating aspect of string theory is the idea of holography: that string theory, which is a theory of quantum gravity, has a dual description as a quantum field theory living on the boundary of the background space. In fact, the converse, that a gauge field theory could have a dual description as a string theory, is an old and important idea in particle physics [10]. There is now one explicit example of such a duality. Maximally supersymmetric four-dimensional Yang-Mills theory is thought to be an exact dual of the IIB superstring on $AdS_5 \times S^5$ background [11, 12, 13]. Moreover, the plane-wave background can be obtained as a Penrose limit of $AdS_5 \times S^5$ and the analogous limit can be taken for Yang-Mills theory to find the Yang-Mills dual of string theory on the plane-wave background [14]. Since superstrings on the plane wave background are more tractable than on $AdS_5 \times S^5$, many interesting aspects of this duality can be studied explicitly. In particular, it gives a promising approach to understanding superstring interactions [15]–[32].

In this paper we will give a brief derivation of the Hagedorn temperature on the plane wave background. We will comment on its relationship with previous work [5, 6, 8, 9]. In particular, we show that the Hagedorn temperature is a monotonically increasing function of the parameter $|f|\sqrt{\alpha'}$ where f is the Ramond-Ramond flux. In the following, we also provide some comments on the interpretation of the Hagedorn behavior in the limit of Yang-Mills theory which is dual to the string theory.

1.1 Hagedorn and AdS/CFT

The AdS/CFT duality asserts that maximally supersymmetric Yang-Mills theory in four dimensions with SU(N) gauge group is exactly dual to type-IIB superstring theory on the background space $AdS_5 \times S^5$ with N units of Ramond-Ramond flux [11]. The radii of curvature of the AdS_5 and S^5 are equal and are given by

$$R = (4\pi g_s N)^{1/4} \sqrt{\alpha'} \,. \tag{1.1}$$

The Yang-Mills and string coupling constants are related by

$$g_{\rm YM}^2 = 4\pi g_s \,. \tag{1.2}$$

This duality gives useful information in limits where either the Yang-Mills or string theory can be analyzed quantitatively. Because of difficulties in quantizing strings in background Ramond-Ramond fields, quantitative results for the string theory on $AdS_5 \times S^5$ are only known in some limits. The first one to be explored is the limit where IIB string theory coincides with classical type-IIB supergravity. This limit is obtained by first taking the classical limit, $g_s \to 0$, holding R constant. This projects onto tree level string theory. Then, it is necessary to take the limit of large string tension. This is done by putting the effective string tension $R^2/\alpha' = \sqrt{4\pi g_s N} \to \infty$. This isolates the lowest energy modes of the string, which are the supergravity fields on the $AdS_5 \times S^5$ background.

On the Yang-Mills side, the first of these limits corresponds to taking $g_{\rm YM}^2 \to 0$ and therefore $N \to \infty$, while holding the 't Hooft coupling

$$\lambda \equiv g_{\rm YM}^2 N \tag{1.3}$$

fixed. This is the 't Hooft large-N (or planar) limit of the gauge theory. For any process, perturbative contributions are the sum of all planar Feynman diagrams — those which can be drawn on a plane without crossing lines. In planar Yang-Mills theory, the effective coupling constant is the 't Hooft coupling λ. Then, the second limit, $R^2/\alpha' \to \infty$, corresponds to taking $\lambda \to \infty$. This gives strongly coupled limit of planar gauge theory.

It is this fact, that a solvable limit of string theory is mapped onto a non-trivial limit of gauge theory which makes the AdS/CFT duality so interesting. At the same time, it makes it difficult to check because reliable computational techniques do not have an overlapping domain of validity. This has limited checks of the conjecture to objects such as two and three point functions of chiral primary operators [33] which do not depend on the coupling constant and thus trivially extrapolate between weak and strong coupling, some anomalies [34, 35] where dependence on the coupling constant is trivial and also to the computations of expectation values of certain Wilson loops [36, 37] and correlators of Wilson loops with chiral primary operators [38]. In the case of a circular loop the Yang-Mills perturbation theory can be summed to all orders in planar diagrams and extrapolated to strong coupling, finding agreement with string theory computations in the supergravity limit.

Recently, another limit of the string and Yang-Mills theory has been studied [14]. A certain limit of the $AdS_5 \times S^5$ string theory background results in the plane-wave metric

$$ds^2 = 2dx^+dx^- - f^2 x_I^2 dx^+dx^+ + dx^I dx^I \,, \qquad I = 1, \ldots, 8 \,, \tag{1.4}$$

and Ramond-Ramond field with non-zero components

$$F_{+1234} = F_{+5678} = 2f \,. \tag{1.5}$$

In the above two equations and throughout this paper we shall use the notation of ref. [4]. In particular, $x^{\pm} = \frac{1}{\sqrt{2}}\left(x^9 \pm x^0\right)$.

Like the supergravity limit, (1.4) is obtained from $AdS_5 \times S^5$ when the curvature is weak and the effective string tension is large, that is, $R^2/\alpha' \to \infty$. However, this limit is taken asymmetrically, in a reference frame which has large angular momentum $J \sim R^2/\alpha'$ on S^5. In this way, the limit retains a particular subset of the the higher level string excitations. Those excitations are described by quantizing the string on the background (1.4), (1.5). This background has the advantage that non-interacting string theory can now be quantized explicitly [3, 4]. For example, the energy spectrum of the non-interacting string is easy to obtain. Further, the interaction terms in the string field hamiltonian have been studied [15, 19, 29, 32].

In the supergravity limit, it is thought that all operators in the Yang-Mills theory that are not protected by supersymmetry get infinitely large conformal dimensions and decouple from the spectrum. The protected operators are just those required to match the classical field degrees of freedom of IIB supergravity linearized about the $AdS_5 \times S^5$ background.

In the plane wave limit, it is still necessary to take $N \to \infty$ and $\lambda \to \infty$ and the conformal dimensions of unprotected Yang-Mills operators become infinite. However, now the operators of interest are those which the AdS/CFT duality maps onto the string states that are found in the limit. Those operators have infinite conformal dimension $\Delta \sim \sqrt{N}$ and also infinite $\mathrm{U}(1) \subset \mathrm{SO}(6)$ R-charge $J \sim \sqrt{N}$. Both diverge as $N \to \infty$ in such a way that the momenta of the corresponding string state, which are identified by [1]

$$p^- \equiv \frac{f}{\sqrt{2}}(\Delta - J)\,, \qquad p^+ \equiv \frac{(\Delta + J)}{\sqrt{2}fR^2}\,, \tag{1.6}$$

remain finite and non-zero [14].

The plane-wave limit of the Yang-Mills theory has two parameters, $g_{\rm YM}$ and the ratio J^2/N which must be held fixed as $N \to \infty$. Combinations of them which appear naturally in the Yang-Mills perturbation theory are

$$\lambda' = \frac{g_{\rm YM}^2 N}{J^2} \tag{1.7}$$

and

$$g_2 = \frac{J^2}{N}\,. \tag{1.8}$$

It was shown in refs. [16, 18] that λ' governs the loop expansion in Yang-Mills theory and it also fixes the distance scale in string theory. Also, the constant g_2 is the effective string coupling in that it governs the loop expansion in string theory. It plays the same role in Yang-Mills theory where it weights Feynman graphs by the genus of the two dimensional surface on which they can be drawn without crossing lines. In light-cone quantization it is natural to consider states which have a fixed light-cone momentum p^+. In this case, we can easily see hat g_2 is related to string loops by using the second equation in (1.6) to trade the Yang-Mills parameters for the pair g_s and p^+, the light-cone momentum of the string,

$$\lambda' = \frac{2}{(f\alpha' p^+)^2} \tag{1.9}$$

$$g_2 = g_{\rm YM}^2 \frac{(f\alpha' p^+)^2}{2} = 2\pi g_s \left(f\alpha' p^+\right)^2\,. \tag{1.10}$$

The free string theory is obtained by putting $4\pi g_s = g_{\rm YM}^2 \to 0$ in conjunction with the large-N limit, with the combination $(f\alpha' p^+)$ non-zero and fixed. This is just the limit where λ' is held constant and g_2 is set to zero. In this limit, all quantities depend on the parameters $g_{\rm YM}$ and N only through the the combination $g_{\rm YM}^2 N = \lambda$, the 't Hooft coupling. This means that free strings are described by the planar limit of Yang-Mills theory. It has now been checked explicitly that the spectrum of free strings is found in the conformal dimensions Δ of certain Yang-Mills operators computed from planar Feynman diagrams [14, 17, 20, 21]. This shows beautiful agreement of the matching between free string states on the plane-wave backrgound and a certain set of operators in the planar limit of Yang-Mills theory.

In this Paper, we shall examine the thermodynamic states of string theory in this limit. We will use the canonical ensemble. The partition function is the trace of the Botlzman distribution,

$$Z(\beta, f) \equiv e^{-\beta F(\beta, f)} = {\rm Tr}\left(e^{-\beta p^0}\right) \tag{1.11}$$

[1]In string theory these are defined by $p^{\pm} = p_{\mp} = \frac{1}{2\pi\alpha'}\int d\sigma\partial_\tau X^{\pm}$ and have a simple form only in the light-cone gauge. In the gauge theory they are defined by the re-scaling of $p_{\pm} = \frac{1}{\sqrt{2}}(\Delta \mp J)$ needed to get the plane-wave limit.

$F(\beta, f)$ is the Helmholtz free energy. Here the trace is over all physical multi-string states. The rest frame energy is given by $p^0 = \frac{1}{\sqrt{2}}(p^+ - p^-)$.

Note that, we could, as was done in ref. [6], introduce a separate parameter for p^+ and p^- and study the theory with two parameters,

$$\tilde{Z}(a, b, f) = \mathrm{Tr}\left(e^{-ap^+ + bp^-}\right). \tag{1.12}$$

However, there is a symmetry of the theory which puts $p^+ \to p^+/\Lambda$, $p^- \to p^-\Lambda$ and $f \to f\Lambda$ which implies that $\tilde{Z}(a, b, f)$ is equal to $\tilde{Z}(\sqrt{ab}, \sqrt{ab}, f\sqrt{a/b})$. Thus, by computing $Z(\beta, f) = \tilde{Z}(\beta/\sqrt{2}, \beta/\sqrt{2}, f)$ we can deduce $\tilde{Z}(a, b, f)$ by identifying $\beta = \sqrt{2ab}$ and replacing $f \to f\sqrt{a/b}$.

For the free string theory, we can compute the Helmholtz free energy in (1.11) exactly. This should then coincide with the free energy of Yang-Mills theory obtained from (1.11) by taking a trace over Yang-Mills states with the momenta identified in (1.6) and where the t'Hooft large-N limit is taken.

In perturbation theory, the free energy would therefore be found as the sum of all orders in planar connected vacuum Feynman diagrams. However, at each order, these diagrams are proportional to N^2 and therefore diverge in the large-N limit. On the other hand, the string theory free energy which we compute is not of order N^2, instead it is of order one. The reason for this discrepancy is that perturbation theory describes the de-confined phase of the gauge theory where the number of physical degrees of freedom is indeed of order N^2, and is only valid if the temperature is greater than the de-confinement transition temperature. That is not the regime described by free strings which rather exist only in the confined phase, found at temperatures below the deconfinement transition and where the number of degrees of freedom is not of order N^2 at large-N, but is of order the number of color singlet operators which, at a given energy, is roughly constant with N. In fact, it is reasonable to identify the Hagedorn temperature, at which a description of the theory by free strings ceases to be meaningful, as the de-confinement transition temperature [39].

At this point, as clarification, we should note that this conformally invariant Yang-Mills theory when it is quantized on $R^3 \times R^1$ does not have a confining phase. It is always in a conformally invariant deconfined phase with a Coulomb-like force law for gauge theory interactions. However, the correct dual of the superstring is Yang-Mills theory with radial quantization, that is, it should be quantized on the space $S^3 \times R^1$ which can be obtianed from $R^3 \times R^1$ by a conformal transformation. It is the energy on the space $S^3 \times R^1$ which is dual to the string energy and is in fact given by the conformal dimension Δ of operators of Yang-Mills theory on the original space $R^3 \times R^1$. From this point of view, the discreteness of the spectrum of Δ comes from the fact that S^3 has finite volume. Further, when Δ is used as the hamiltonian, the finite temperature Yang-Mills theory lives on the space $S^3 \times S^1$ where the time direction is euclidean and has been periodically identified, $X^0 \sim X^0 + \beta$, with the appropriate antiperiodic boundary condition for fermions.

Even on this space, since the volume is finite, one does not expect a confinement-deconfinement phase transition when N is finite. This transition could only occur at infinite N. However, it is just the infinite N limit that must be taken to obtain strings on the plane-wave background. In this limit, the Yang-Mills theory could have a phase transition corresponding to confinement-deconfinement as the temperature is varied. An order parameter for such a phase transition is the Polyakov loop [40, 41]

$$\left\langle \mathrm{Tr}\mathcal{P}e^{i\oint_{S^1}A}\right\rangle$$

which, in this adjoint gauge theory, transforms under a certain discrete large gauge symmetry related to confinement. There are many examples of gauge theories where this order parameter can be explicitly seen to characterize confinement [42]–[45]. For example, the one-dimensional non-abelian coulomb gas studied in refs. [46, 47] has a deconfiment transition only at infinite N, corresponding to a re-arrangement of the distribution of eigenvalues of the unitary matrix $\mathcal{P}e^{i\oint_{S^1}A}$, analogous to that which is well known to occur at large-N in unitary matrix models [48]. If, as is suggested in ref. [39], the de-confinement and Hagedorn behaviors can be identified, the existence of a Hagedorn temperature in the string theory dual is a confirmation of the existence of a de-confinement transition in the Yang-Mills theory, at least in the planar limit which is dual to free strings.

We shall indeed find that, in the limit where the string coupling g_s is put to zero, there is a Hagedorn temperature for all finite values of the parameter f of the background, implying that the planar Yang-Mills theory indeed has a confinement-deconfinement phase transition. The string theory analysis gives the value of the transition temperature for the gauge theory.

It is interesting to contrast the situation of the plane-wave background to that in AdS/CFT before the plane wave limit is taken. In the latter case, the Hagedorn spectrum for the operator Δ appears in Yang-Mills theory as the exponentially increasing multiplicity of an infinite tower of operators which are gauge invariant traces of local products of the fields. When N is infinite, products of all sizes are independent operators. When the 't Hooft coupling λ is small and Δ of these operators deviates little from the tree level values, the number of operators with a given value of Δ can be counted [49] and it indeed grows exponentially with increasing Δ, producing a Hagedorn spectrum for Yang-Mills theory quantized on $S^3\times R^1$. Thus, we would expect large-N Yang-Mills theory to have a Hagedorn temperature if λ is small enough. When λ gets large, the anomalous dimensions of operators get large and they begin to decouple from the low-lying spectrum.

At very large λ the dynamics is that of classical supergravity, perhaps with stringy corrections which are suppressed by factors of $1/\sqrt{\lambda}$. It is known that supergravity with an asymptotically AdS geometry has a Hawking-Page phase transition [50] between an AdS black hole state, which can be interpreted as the de-confined phase, and one which is AdS space with periodic euclidean time, which can be interpreted as the confined phase. Indeed, the fact that the free energy of the black hole is of order N^2, whereas in the periodic AdS space it is of order one is in line with this interpretation [13, 51]. One could then speculate that the Hagedorn behavior which is seen in weakly coupled planar Yang-Mills theory evolves to the Hawking-Page transition of supergravity with periodic euclidean time as λ goes from zero to infinity, and further that this corresponds to the de-confinement phase transition. It is also clear that the temperature where the Hawking-Page transition occurs is proportional to the radius of curvature of the AdS space, $T_H \sim R/\alpha' \sim \lambda^{1/4}$ and it actually becomes large as $\lambda \to \infty$, as we expect.

In contrast, the partition function of the limit of Yang-Mills theory which corresponds to the plane wave background would be the trace over states of the exponential of the operator

$$Z = \mathrm{Tr}\exp\left(-\frac{\beta^2}{\alpha'}\frac{(\Delta+J)}{2\beta f\sqrt{\lambda}} - \beta f\frac{\Delta - J}{2}\right). \tag{1.13}$$

We see that the parameters indeed appear naturally in the combinations β^2/α' and βf.

1.1.1 Large-f

In the limit where f is large for fixed β and λ, the states which dominate the partition sum are those with $\Delta = J$. They are just the single and multi-trace chiral primary operators, $\mathrm{Tr}(Z_1^J)\mathrm{Tr}(Z_2^J)\ldots\mathrm{Tr}(Z_k^J)$ whose conformal dimensions $\Delta = \sum J_i = J$ are protected by supersymmetry. They correspond to single and multi-string states where the string is in its lowest state, the string state which is described by the supersymmetric vacuum of the worldsheet sigma model.

To find the partition functions, we can think of the number of times J_1 appears in the product of traces as the occupation number n_{J_1}, the number of strings which are in the state with quantum number J_1. J_1 can have both integers and half integers values. The contribution of this state to total J is $J_1 n_{J_1}$. We enforce Bose statistics by summing over all occupation numbers of all states, to get the partition function

$$Z = e^{-\beta F} = \prod_{J=1/2,1,3/2\ldots}^{\infty}\sum_{n_J=0}^{\infty}\exp\left(-\frac{\beta n_J J}{\alpha' f\sqrt{\lambda}}\right) = \prod_{J=1/2,1,3/2\ldots}^{\infty}\frac{1}{1-e^{-\frac{\beta J}{\alpha' f\sqrt{\lambda}}}} \tag{1.14}$$

$$F = \frac{1}{\beta}\sum_{J=1/2,1,3/2\ldots}^{\infty}\ln\left(1-e^{-\frac{\beta J}{\alpha' f\sqrt{\lambda}}}\right) = -\frac{1}{\beta}\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\frac{n\beta p}{2\alpha' f\sqrt{\lambda}}} \tag{1.15}$$

$$= -\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{e^{\frac{n\beta}{2\alpha' f\sqrt{\lambda}}}-1} = -\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{2\alpha' f\sqrt{\lambda}}{\beta n^2} = -\frac{\pi^2\alpha' f\sqrt{\lambda}}{3\beta^2} \tag{1.16}$$

where, $p = 2J$ and, in the last step, we have taken the large f limit. We will see that this coincides with the large f limit of the string partition function which we will find in the following section. To do this, we need to identify the infinite length of the X^--direction. We can do this by examining the quantization of P^+. J and Δ can be integers and, for fermions, half-integers, but in all cases, the sum $\Delta + J$ are integers. Consequently, P^+ should be of the form $\sqrt{2}\pi \cdot \mathrm{integer}/L$. From this we identify the infinite length in the X^9-direction as $L = 2\pi\alpha' f\sqrt{\lambda}$. Then the large f limit in (1.16) is

$$F \to -\frac{\pi L}{6\beta^2}\,.$$

One could speculate about what happens when string interactions are switched on. In the asymptotically AdS space, which is dual to Yang-Mills theory with finite J, string interactions are restored by relaxing the large-N limit. This produce a cutoff of order N^2

on the number of independent traces of local operators, and therefore it should cut off the Hagedorn behavior — at least the counting of independent operators in weakly coupled Yang-Mills theory no longer produces a Hagedorn spectrum. Commensurate with this, we do not expect a de-confinement phase transition in Yang-Mills theory in the finite volume of $S^3 \times S^1$ if N is finite. This makes the prediction that interacting strings do not have a finite temperature phase transition on an asymptotically AdS space.

On the other hand, to obtain the plane-wave background, we should always take the limit $N \to \infty$. This would suggest that we always have a Hagedorn spectrum of traces of local operators, the main question being whether their quantum number $\Delta - J$ remains finite when both coupling constants, λ' and g_2 are non-zero. It is known that when $\Delta - J$ depends on g_2 it is shifted by a small amount when g_2 is small [22, 24]. Thus, we can speculate that, as long as g_2 is small enough, the Hagedorn behavior indeed persists when string interactions are present.

1.2 Other issues

In the Yang-Mills partition function which is (1.11) with the momenta (1.6), we should take $R/\sqrt{\alpha'} \to \infty$ while holding the temperature $\beta/\sqrt{\alpha'}$ fixed. The states which contribute in the trace are those which have finite $\Delta - J$. On the other hand, $\Delta + J$, and therefore both Δ and J get arbitrarily large as $R/\sqrt{\alpha'} \to \infty$. One might question whether this limit is sensible. In the usual limit, p^+ and p^- are held constant when N is taken to infinity. Here, instead, the temperature is held constant and it is not a priori clear that holding the temperature constant and finite actually samples the states of the Yang-Mills theory which coincide with the string states. It would be interesting to find a way to check this directly. Unfortunately, the standard perturbative computation using path integrals is only valid in the de-confined phase which occurs at high temperatures where the confined states that we find in string theory would be difficult to detect.

An important issue is the possible existence of zero modes of p^+. Any protected operator for which $J^2/N \to 0$ as $N \to \infty$ are zero modes of p^+. Some of these are just at the $p^+ = 0$ edge of the continuum spectrum and are included in our analysis. These are the operators $\mathrm{Tr}Z^J$ where J is not taken to infinity fast enough as N is taken to infinity. There are also other operators, such as the protected operators in the dilaton supermultiplet which have finite non-zero $\Delta - J$ and for which $\Delta + J$ are finite in the limit as $N \to \infty$, so that $p^+ = 0$. These could be considered as discrete zero modes of p^+ which seem to have no analog in the light-cone string spectrum. This would seem to be a miss-match between the string and Yang-Mills spectra.

Recently, there has been some discussion on the Hagedorn behavior of pp-wave strings [5, 6] also in discrete light cone quantization [7]. It is well known that when string theories are placed in a background electric NS B field or in a metric, the Hagedorn temperature depends on the parameters of the background [52, 53, 54]. Here we shall find that also the RR flux (1.5) felt by a string in the pp-wave metric modifies the Hagedorn temperature. We shall also clarify some of the issues related to the small and large f limit of the Hagedorn temperature, providing results that even if in qualitative agreement with those of refs. [5, 6] differ quantitatively. We shall then study the thermodynamic behavior of strings in geometries that arise in D1-D5 systems as $AdS_3 \times S^3 \times T^4$ with NS-NS and RR 3-form backgrounds [55, 14, 7]. It would be intersting to rederive our results by means of a path integral procedure and then generalize them to higher genera [56].

2. Free energy

The free energy of a gas of noninteracting superstrings is given by summing the free energies of free particles over all of the particle species in the string spectrum.[2] Each boson in the spectrum contributes

$$F = \frac{1}{\beta}\mathrm{Tr}\ln\left(1 - e^{-\beta p^0}\right) = -\sum_{n=1}^{\infty}\frac{1}{n\beta}\mathrm{Tr}e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)}\,. \tag{2.1}$$

where p^0 and $p^{\pm}$ are the energy and light-cone momenta of the particle. Similarly, each fermion contributes

$$F = -\frac{1}{\beta}\mathrm{Tr}\ln\left(1 + e^{-\beta p^0}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n\beta}\mathrm{Tr}e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)}\,. \tag{2.2}$$

We emphasize that the trace in each case is over the spectrum of single particle states, rather than multi-particle states. The total free energy is given by summing (2.1) and (2.2) over the particles which appear in the string spectrum.

In the usual Minkowski background superstring, in the NSR formulation, after the GSO projection the number of fermions and bosons at all levels is the same. The GSO projection is necessary for the theory to be supersymmetric. If one did not GSO project, then the tachyon is unpaired, then again the theory does not have supersymmetry. In the GS formulation of the superstring theory one does not need to GSO project and the fermion zero modes of the worldsheet theory make the degeneracies that give every level the same numbers of bosons and fermions [58, 59].

For strings propagating on a pp-wave background, things are slightly different. Because of supersymmetry, most of the string spectrum has paired fermionic and bosonic states, so for these states we can take the average of the two expressions,

$$F_{\mathrm{susy}} = -\sum_{n=1}^{\infty}\frac{1-(-1)^n}{2n\beta}\mathrm{Tr}e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)}\,. \tag{2.3}$$

However, there is one set of states in the spectrum which will turn out not to have a superpartner, they are the lowest energy excitations which are bosons and have vanishing light-cone hamiltonian p^- and arbitrary p^+. To take these bosons into account (2.3) must be amended to read

$$F = -\sum_{n=1,\mathrm{odd}}^{\infty}\frac{1}{n\beta}\mathrm{Tr}_{(p^-<0)}e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)} - \sum_{n=1}^{\infty}\frac{1}{n\beta}\mathrm{Tr}_{(p^-=0)}e^{-\frac{n\beta}{\sqrt{2}}(p^+ - p^-)}\,. \tag{2.4}$$

[2]For a derivation of this formula, see ref. [57].

Summing these operators over the spectrum of the operators p^- and p^+ which are found in light-cone quantization of the string should yield the free energy. The last term is easily evaluated by noting that the measure for the trace over p^+ is $\frac{L}{\sqrt{2\pi}}\int_0^\infty dp^+$, where L is the (infinite) length of the 9'th dimension. We combine the odd integer sum in the last term with the first term. This removes the constraint on the spectrum in that term. Then,

$$F = -\sum_{n=1,\mathrm{odd}}^{\infty}\frac{1}{n\beta}\mathrm{Tr}e^{-\frac{n\beta}{\sqrt{2}}(p^+-p^-)} - \frac{L}{\pi\beta^2}\sum_{n=2,\mathrm{even}}^{\infty}\frac{1}{n^2} = -\sum_{n=1,\mathrm{odd}}^{\infty}\frac{1}{n\beta}\mathrm{Tr}e^{-\frac{n\beta}{\sqrt{2}}(p^+-p^-)} - \frac{L\pi}{24\beta^2}\,. \tag{2.5}$$

To proceed, we must examine the string spectrum.

The Green-Schwarz IIB-superstring can be quantized in the light-cone gauge. The explicit form of the light-cone hamiltonian is

$$\begin{aligned} H &\equiv -P^- \\ &= f\left(a_0^I\bar{a}_0^I + 2\bar{\theta}_0\bar{\gamma}^-\Pi\theta_0 + 4\right) + \frac{1}{\alpha' p^+}\sum_{\mathcal{I}=1,2}\sum_{m=1}^{\infty}\sqrt{m^2+(\alpha' p^+ f)^2}\left(a_m^{\mathcal{I}I}\bar{a}_m^{\mathcal{I}I} + \eta_m^{\mathcal{I}}\bar{\gamma}^-\bar{\eta}_m^{\mathcal{I}}\right) \\ &= f\left(N_0^B + N_0^F + 4\right) + \frac{1}{\alpha' p^+}\sum_{\mathcal{I}=1}^{2}\sum_{m=1}^{\infty}\sqrt{m^2+(\alpha' p^+ f)^2}\left(N_{\mathcal{I}m}^B + N_{\mathcal{I}m}^F\right). \end{aligned} \tag{2.6}$$

where we refer to [4] for the notation.

The level matching condition $N_1 = N_2$ also has to be enforced by introducing an integration over the Lagrange multiplier τ_1. Explicitly (2.5) means

$$\begin{aligned} F = &-\sum_{n=1,\mathrm{odd}}^{\infty}\frac{L}{4\pi^2\alpha'}\int_0^\infty\frac{d\tau_2}{\tau_2^2}\prod_{I=1}^{8}\sum_{N_0^{B,I}=0}^{\infty}\sum_{n_R,n_L=0}^{4}\sum_{N_{1,2m}^{B,I}=0}^{\infty}\sum_{N_{1,2m}^{F}=0}^{8}\times \\ &\times\int_{-\frac{1}{2}}^{\frac{1}{2}}d\tau_1 e^{2\pi i\tau_1\sum_{m=1}^{\infty}m(N_{1m}^{B,I}+N_{1m}^F-N_{2m}^{B,I}-N_{2m}^F)}\times \\ &\times e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}}e^{-\frac{n\beta f N_0^{B,I}}{\sqrt{2}}}\binom{4}{n_R}\binom{4}{n_L}e^{-\frac{n\beta f}{\sqrt{2}}(-n_R+n_L+4)}\times \\ &\times\binom{8}{N_{1m}^F}\binom{8}{N_{2m}^F}e^{-\sum_{m=1}^{\infty}R_m(N_{1m}^{B,I}+N_{1m}^F+N_{2m}^{B,I}+N_{2m}^F)} - \frac{L\pi}{24\beta^2} \end{aligned} \tag{2.7}$$

where L is the length of the longitudinal direction, $N_0^F = -n_R + n_L$ and

$$R_m = 2\pi\tau_2\sqrt{m^2+\mu^2}\,,\qquad \tau_2 = \frac{n\beta}{2\sqrt{2}\pi\alpha' p^+}\,,\qquad \mu = \alpha' p^+ f = \frac{n\beta f}{2\sqrt{2}\pi\tau_2}\,, \tag{2.8}$$

Due to the anticommutation relations of the creation-annihilation fermion operators, the degeneracy of a state with $n_{R,L}$ fermions is given by the binomial coefficient $\binom{4}{n_{R,L}}$. Analogously the occupation number $N_{\mathcal{I}m}^F$ for the fermion non-zero modes, which have eight independent components, runs from 0 to 8 and the degeneracy is given by the binomial coefficient $\binom{8}{N_{\mathcal{I}m}^F}$. Summing over the zero-modes, the free energy can be written as

$$F = -\sum_{n=1,\mathrm{odd}}^{\infty}\frac{L}{4\pi^2\alpha'}\int_0^\infty\frac{d\tau_2}{\tau_2^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}d\tau_1 e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}}\left(\frac{1}{1-e^{-\frac{n\beta f}{\sqrt{2}}}}\right)^8\times$$

$$\times e^{-\frac{4n\beta f}{\sqrt{2}}}\left(1+e^{-\frac{n\beta f}{\sqrt{2}}}\right)^4\left(1+e^{\frac{n\beta f}{\sqrt{2}}}\right)^4\left|G\left(\tau_1,\tau_2,\frac{n\beta f}{\sqrt{2}2\pi\tau_2}\right)\right|^2-\frac{L\pi}{24\beta^2} \tag{2.9}$$

G is the function

$$G(\tau_1,\tau_2,\mu)=\prod_{I=1}^{8}\sum_{N_{1m}^{B,I}=0}^{\infty}\sum_{N_{1m}^{F}=0}^{8}\begin{pmatrix}8\\N_{1m}^F\end{pmatrix}e^{2\pi i\tau_1\sum_{m=1}^{\infty}m(N_{1m}^{B,I}+N_{1m}^F)}e^{-\sum_{m=1}^{\infty}R_m(N_{1m}^{B,I}+N_{1m}^F)}\,. \tag{2.10}$$

Performing the sums over the occupation numbers the generating function becomes

$$G(\tau_1,\tau_2,\mu)=\prod_{m=1}^{\infty}\left(\frac{1+e^{-2\pi\tau_2\sqrt{m^2+\mu^2}+2\pi i\tau_1 m}}{1-e^{-2\pi\tau_2\sqrt{m^2+\mu^2}+2\pi i\tau_1 m}}\right)^8 \tag{2.11}$$

so that the free energy reads

$$F=-\sum_{n=1,\mathrm{odd}}^{\infty}\frac{L}{4\pi^2\alpha'}\int_0^{\infty}\frac{d\tau_2}{\tau_2^2}\int_{-\frac{1}{2}}^{\frac{1}{2}}d\tau_1 e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}}\prod_{m=-\infty}^{\infty}\left(\frac{1+e^{-2\pi\tau_2\sqrt{m^2+\mu^2}+2\pi i\tau_1 m}}{1-e^{-2\pi\tau_2\sqrt{m^2+\mu^2}+2\pi i\tau_1 m}}\right)^8-\frac{L\pi}{24\beta^2}\,. \tag{2.12}$$

This equation is **not** in agreement with eq.(3.3) of ref. [5], because it differs by the contribution of the zero light-cone energy mode. The limit $f\to\infty$ of (2.12) can be easily computed since $G(\tau_1,\tau_2,\mu)\to 1$ in this limit so that F becomes

$$F=-\frac{\pi L}{6\beta^2} \tag{2.13}$$

which coincides with the free energy of the dual gauge theory (1.16) computed in this limit in the introduction. Note that this is the free energy density of a gas of massless particles in two dimensions. Indeed, the lowest energy states of the string are massless chiral bosons which propagate down the two spacetime dimensional axis of the pp-wave space made of the X^+ and X^- directions. Further they are chiral, in that their spectrum is composed entirely of left-moving particles. The spectrum of these particles is protected by supersymmetry, so we expect that this limit of the partition function is not corrected by string interactions.

We shall now extract information directly from (2.12) instead of turning to the path integral approach as in [5]. To compute the Hagedorn temperature we need to estimate the asymptotic behavior of the product in (2.12). This will be crudely estimated in this section; a more precise estimate will be obtained in the next section by using its modular transformations properties [60]. Consider the function defined by

$$Z(\tau_1,\tau_2,\mu)\equiv\prod_{m=-\infty}^{\infty}\left(\frac{1+e^{-2\pi\tau_2\sqrt{m^2+\mu^2}+2\pi i\tau_1 m}}{1-e^{-2\pi\tau_2\sqrt{m^2+\mu^2}+2\pi i\tau_1 m}}\right). \tag{2.14}$$

It diverges only when τ_1, τ_2 and βf vanish, let us then consider these limits by taking first $\tau_1=0$ and then $\tau_2\to 0$. For $\tau_1=0$ it reads

$$Z(0,\tau_2,\mu)=\exp\left\{\sum_{m=-\infty}^{\infty}\ln\left(\frac{1+e^{-2\pi\tau_2\sqrt{m^2+\mu^2}}}{1-e^{-2\pi\tau_2\sqrt{m^2+\mu^2}}}\right)\right\}$$

$$= \exp\left\{ -\sum_{m=-\infty}^{\infty}\sum_{p=1}^{\infty}\left[\frac{(-1)^p}{p}-\frac{1}{p}\right] e^{-2\pi\tau_2 p\sqrt{m^2+\mu^2}}\right\}. \tag{2.15}$$

Using the integral identity [61]

$$e^{-2\sqrt{ab}}\frac{1}{2}\sqrt{\frac{\pi}{a}} = \int_0^\infty e^{-at^2-\frac{b}{t^2}}dt\,, \qquad (a,b>0) \tag{2.16}$$

one can write

$$Z = \exp\left\{ +2\sum_{m=-\infty}^{\infty}\sum_{p_{odd}=1}^{\infty}\frac{1}{p}\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2-\frac{\pi^2\tau_2^2p^2m^2}{t^2}-\frac{n^2\beta^2f^2p^2}{8t^2}}dt\right\}. \tag{2.17}$$

In the limit of interest $\tau_2 \to 0$, the sum over m may be approximated by an integral $\sum_m \simeq \int_{-\infty}^{\infty} dm$. The integration over m is gaussian and can be performed. The leading behavior in the $\tau_2 \to 0$ limit then is

$$Z \simeq \exp\left\{ \frac{n\beta f}{\sqrt{2}\pi\tau_2}\sum_{p=1}^{\infty}\frac{[1-(-1)^p]}{p}K_1\left(\frac{n\beta f p}{\sqrt{2}}\right)\right\} \tag{2.18}$$

where K_1 is the modified Bessel function. Using the series expansion of the Bessel function it is easy to see that the leading term in the limit $\beta f \to 0$ (2.18) reproduces the expected flat space behavior. A more precise derivation of this result will be obtained in the next section.

3. Modular properties of Z

Consider the function defined by

$$Z_{a,b}(\tau_1,\tau_2,x) = \prod_{m=-\infty}^{\infty}\left(1-e^{-2\pi\tau_2\sqrt{x^2+(m+b)^2}+2\pi i\tau_1(m+b)+2\pi i a}\right). \tag{3.1}$$

The partition function (2.14) is given by the ratio

$$Z\left(\tau_1,\tau_2,\frac{n\beta f}{\sqrt{2}2\pi\tau_2}\right) = \frac{Z_{\frac{1}{2},0}(\tau_1,\tau_2,\frac{n\beta f}{2\pi\sqrt{2}\tau_2})}{Z_{0,0}(\tau_1,\tau_2,\frac{n\beta f}{2\pi\sqrt{2}\tau_2})}. \tag{3.2}$$

It will turn out to be useful to define

$$\Delta_b(x) = -\frac{1}{2\pi^2}\sum_{p=1}^{\infty}\cos(2\pi bp)\int_0^\infty ds\; e^{-p^2s-\frac{\pi^2x^2}{s}} = -\frac{x}{\pi}\sum_{p=1}^{\infty}\frac{\cos(2\pi bp)}{p}K_1(2\pi xp)\,. \tag{3.3}$$

The quantity $\Delta_b(x)$ corresponds to the zero-energy (Casimir energy) of a 2D complex scalar boson ϕ of mass m with the twisted boundary condition $\phi(\tau,\sigma+\pi) = e^{2\pi i b}\phi(\tau,\sigma)$. In the massless limit this zero energy correctly reproduces the familiar value

$$\lim_{x\to 0}\Delta_b(x) = \frac{1}{24}-\frac{1}{8}(2b-1)^2\,. \tag{3.4}$$

Following the appendix A of ref. [60] it is not difficult to derive the modular property of (3.1)

$$\ln Z_{a,b}(\tau_1,\tau_2,x) = \ln Z_{-b,a}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{x}{|\tau|}\right) - 2\pi\tau_2\Delta_b(x) + 2\pi\frac{\tau_2}{|\tau|^2}\Delta_a\left(\frac{x}{|\tau|}\right). \tag{3.5}$$

As a consequence the transformation properties of (2.14) are

$$\begin{aligned}\ln Z\left(\tau_1,\tau_2,\frac{n\beta f}{\sqrt{2}2\pi\tau_2}\right) &= \ln Z_{0,\frac{1}{2}}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{n\beta f|\tau|}{2\pi\sqrt{2}\tau_2}\right) - \ln Z_{0,0}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{n\beta f|\tau|}{2\pi\sqrt{2}\tau_2}\right) + \\ &+2\pi\frac{\tau_2}{|\tau|^2}\left[\Delta_{\frac{1}{2}}\left(\frac{n\beta f|\tau|}{2\pi\sqrt{2}\tau_2}\right) - \Delta_0\left(\frac{n\beta f|\tau|}{2\pi\sqrt{2}\tau_2}\right)\right]. \end{aligned} \tag{3.6}$$

From the definition of the Casimir energy (3.3) the last two terms in (3.6) read

$$\frac{n\beta f}{\sqrt{2}\pi|\tau|}\sum_{p=1}^{\infty}\frac{[1-(-1)^p]}{p}K_1\left(\frac{n\beta f p|\tau|}{\sqrt{2}\tau_2}\right). \tag{3.7}$$

In the limit $\tau_1 \to 0$ and $\tau_2 \to 0$ the first two terms in (3.6) behave smoothly whereas the second two give precisely the behavior found in (2.18).

4. Hagedorn temperature

The asymptotic value of the free energy (2.9) then is

$$F \sim \sum_{n=1}^{\infty}\frac{(-1)^n-1}{8\pi^2\alpha'}L\int_0^{\infty}\frac{d\tau_2}{\tau_2^2}e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}}\exp\left\{\frac{8n\beta f}{\sqrt{2}\pi\tau_2}\sum_{p=1}^{\infty}\frac{[1-(-1)^p]}{p}K_1\left(\frac{n\beta f p}{\sqrt{2}}\right)\right\}. \tag{4.1}$$

The biggest value of β for which this expression diverges in the $\tau_2 \to 0$ limit is obtained by taking the $n = 1$ mode. When the exponent in the integrand of (4.1) vanishes, F starts to diverge so that the Hagedorn temperature is defined by the equation

$$\frac{\beta_H^2}{4\pi\alpha'} = \frac{8\beta_H f}{\sqrt{2}\pi}\sum_{p=1}^{\infty}\frac{[1-(-1)^p]}{p}K_1\left(\frac{\beta_H f p}{\sqrt{2}}\right). \tag{4.2}$$

Taking the derivative with respect to f one gets

$$\frac{\partial\beta_H}{\partial f} = -\frac{8\alpha'|f|\beta_H\sum_{p=1}^{\infty}[1-(-1)^p]K_0\left(\frac{\beta_H f p}{\sqrt{2}}\right)}{1+8\alpha' f^2\sum_{p=1}^{\infty}[1-(-1)^p]K_0\left(\frac{\beta_H f p}{\sqrt{2}}\right)}. \tag{4.3}$$

The r.h.s. of this equation is always negative thus β_H is a decreasing function of $|f|\sqrt{\alpha'}$ and consequently T_H is an increasing function of $|f|\sqrt{\alpha'}$.

We shall now study the behavior of equation (4.2) in the small and large f limit. For small f it is necessary to rewrite it as a power series in βf and then solve for β. This will be rigorously done in the the next section and it will allow us to derive the correct result for the Hagedorn temperature at small f. For large f the behavior of (4.2) it is much easier to extract and it should reproduce the dual gauge theory behavior.

4.1 Expansion for small f

To rewrite (4.2) as a series expansion in f, we shall use the Mellin transform procedure. The series

$$S_b(x) = \sum_{p=1}^{\infty} \frac{1}{p} K_1(xp) \tag{4.4}$$

can in fact be rewritten as a power series in x by means of a Mellin transformation. The Mellin transform of $S_b(x)$ reads

$$M(s) = \int_0^\infty dx x^{s-1} S_b(x) = \sum_{p=1}^{\infty} \int_0^\infty dx \int_0^\infty \frac{dt}{4t^2} x^s \, e^{-t-\frac{x^2p^2}{4t}} . \tag{4.5}$$

Changing the integration variable x to $y = x^2p^2/(4t)$, $M(s)$ becomes

$$M(s) = \sum_{p=1}^{\infty} \int_0^\infty \frac{dy}{8} \left(\frac{2}{p}\right)^{s+1} y^{(s-1)/2} e^{-y} \int_0^\infty \frac{dt}{t^2} t^{(s+1)/2} e^{-t} . \tag{4.6}$$

The Mellin transform $M(s)$ exists provided the integrals over y and t are bounded for some $s > k$ with $k > 0$. In our case the integrals can be done for $s > 1$ and $M(s)$ is

$$M(s) = 2^{s-2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \zeta(s+1) . \tag{4.7}$$

The inversion of the Mellin transform gives back the function $S_b(x)$ and is accomplished by means of the inversion integral

$$S_b(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} ds M(s) x^{-s} \tag{4.8}$$

where $C > k = 1$. The integral is well defined and to compute it we must close the contour and use the residue theorem. For this purpose it is convenient to change the argument of $\zeta(s+1)$ in the integrand as [61]

$$\zeta(s+1) = \pi^{s+1/2} \frac{\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \zeta(-s) . \tag{4.9}$$

Therefore

$$S_b(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} ds \left(\frac{2\pi}{x}\right)^s \frac{\sqrt{\pi}}{4} \Gamma\left(-\frac{s}{2}\right) \Gamma\left(\frac{s-1}{2}\right) \zeta(-s) . \tag{4.10}$$

The contour can now be closed on the left so that the poles are at $s = 1, 0, -1, 1-2k, \ldots$ for $k = 2, 3, \ldots$. The residues can be easily computed and the result is

$$\begin{aligned} S_b(x) &= \frac{\pi^2}{6x} - \frac{\pi}{2} + \frac{x}{8}\left(1 - 2\gamma + 2\ln\frac{4\pi}{x}\right) \\ &\quad + \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x}{2\pi}\right)^{2k-1} \frac{\sqrt{\pi}}{2} \Gamma\left(k - \frac{1}{2}\right) \zeta(2k-1) \end{aligned} \tag{4.11}$$

where γ is the Euler constant. Analogously one can rewrite the series

$$S_f(x) = \sum_{p=1}^{\infty} \frac{(-1)^p}{p} K_1(xp) \tag{4.12}$$

as

$$\begin{aligned} S_f(x) &= -\frac{\pi^2}{12x} + \frac{x}{8}\left(1 - 2\gamma + 2\ln\frac{\pi}{x}\right) + \\ &+ \sum_{k=2}^{\infty} \frac{(-1)^k}{k!}(2^{2k-1} - 1)\left(\frac{x}{2\pi}\right)^{2k-1} \frac{\sqrt{\pi}}{2}\Gamma\left(k - \frac{1}{2}\right)\zeta(2k-1)\,. \end{aligned} \tag{4.13}$$

The series appearing in the formula for the Hagedorn temperature (4.2) can then be rewritten as

$$\begin{aligned} S_b(x) - S_f(x) &= \sum_{p=1}^{\infty} \frac{1 - (-1)^p}{p} K_1(xp) \\ &= \frac{\pi^2}{4x} - \frac{\pi}{2} + \frac{x}{2}\ln 2 - \\ &- \sum_{k=2}^{\infty} \frac{(-1)^k}{k!}(2^{2k-1} - 2)\left(\frac{x}{2\pi}\right)^{2k-1} \frac{\sqrt{\pi}}{2}\Gamma\left(k - \frac{1}{2}\right)\zeta(2k-1)\,. \end{aligned} \tag{4.14}$$

Using these results for the series difference in (4.2) one can derive the following formula for the Hagedorn temperature in the limit of small f.

$$\frac{\beta_H^2}{4\pi\alpha'} = 2\pi - \frac{4\beta_H f}{\sqrt{2}} + \frac{2\beta_H^2 f^2 \ln 2}{\pi} - \sum_{k=2}^{\infty} \frac{(-1)^k (2^{2k} - 4) 4\sqrt{\pi}}{k!} \left(\frac{\beta_H f}{2\pi\sqrt{2}}\right)^{2k} \Gamma(k - \frac{1}{2})\zeta(2k-1)\,. \tag{4.15}$$

Keeping only the two leading terms in the expansion of (4.15) we get

$$\beta_H^2(1 - 8\alpha' f^2 \ln 2) = 8\pi^2\alpha' - \frac{16\pi\alpha'\beta_H f}{\sqrt{2}}\,. \tag{4.16}$$

The Hagedorn temperature then is

$$T_H = \frac{1}{2\pi\sqrt{2\alpha'}}\left(1 + 2\sqrt{\alpha'}f + 2(1 - 2\ln 2)\alpha' f^2\right). \tag{4.17}$$

As in [5] the Hagedorn temperature increases for small values of $f^2\alpha'$ but the second term differs from the one derived in [5] by a factor of $4\pi\sqrt{2}$.

In the flat space limit $f \to 0$ we recover the well known superstring Hagedorn temperature

$$T_H = \frac{1}{\beta_H} = \frac{1}{2\pi\sqrt{2\alpha'}}\,. \tag{4.18}$$

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4.2 The large f limit

Let us now consider the large f behavior of eq. (4.2). It is particularly interesting to examine this limit because it is in this limit pp-wave that type-IIB string theory is supposed to be dual to a subsector of a particular Yang-Mills theory [14]. For large value of f, the most relevant contribution to the series of the modified Bessel function K_1 is given by taking $p = 1$ in (4.2). For large value of its argument the Bessel function in fact can be approximated by

$$K_1\left(\frac{\beta_H f p}{\sqrt{2}}\right) \sim \sqrt{\frac{\pi}{\sqrt{2}\beta_H f p}}\exp\left(-\frac{\beta_H f p}{\sqrt{2}}\right) \tag{4.19}$$

so that terms with higher values of p are exponentially suppressed. The equation (4.2) for the Hagedorn temperature becomes

$$\frac{\beta_H^2}{4\pi\alpha'} = 8\sqrt{\frac{\beta_H f\sqrt{2}}{\pi}}\exp\left(-\frac{\beta_H f}{\sqrt{2}}\right) \to_{f\to\infty} 0\,. \tag{4.20}$$

The rapid vanishing of the Bessel function in the large f limit implies that the Hagedorn temperature increases with f and for very large f is pushed toward infinity. This means that in this regime there is no Hagedorn transition at any finite temperature but instead the Hagedorn temperature is a limiting temperature. This is expected since the large f limit should indeed reproduce the gauge theory behavior.

5. $AdS_3 \times S^3$ in NS-NS and RR 3-form backgrounds

The limit that gives the metric (1.4) in the $AdS_5 \times S^5$ geometry can be taken also in other geometries. As a particular case one can consider the $AdS_3 \times S^3$ geometry [62, 55, 14]. In this case the radii of AdS_3 and S^3 are the same and the computation is identical to the one we did above for $AdS_5 \times S^5$. It is interesting to consider a situation with a mixture of NS-NS and RR 3-form field strengths. The six dimensional plane-wave metric is

$$\begin{aligned} ds^2 &= 2dx^+dx^- - f^2\vec{y}^2 dx^+dx^+ + d\vec{y}^2 && (5.1)\\ F^{NS}_{+12} &= F^{NS}_{+34} = C_1 f\cos\alpha \\ F^{RR}_{+12} &= F^{R}_{+34} = C_2 f\sin\alpha && (5.2)\\ & && (5.3) \end{aligned}$$

where $\vec{y}$ parametrizes a point on T^4 and α is a fixed parameter which allows us to interpolate between the purely NS background $\alpha = 0$ and the purely RR background $\alpha = \pi/2$. C_1 and C_2 are constants depending on the string coupling and the normalization of the NS and RR field strenghts. In addition to the six coordinates in (5.1) we have four additional directions which we can take to be T^4.

The light-cone hamiltonian is

$$H = \sum_{n=-\infty}^{\infty} N_n\sqrt{f^2\sin\alpha^2 + \left(f\cos\alpha + \frac{n}{\alpha' p^+}\right)^2} + 2\frac{L_0^{T^4} + \bar{L}_0^{T^4}}{\alpha' p^+} \tag{5.4}$$

where the first term takes into account the massive bosons and fermions and the second term takes into account the massless bosons and fermions.

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The computation of the free energy is similar to the one we performed in the previous section for the $AdS_5 \times S^5$ geometry. It reads

$$F = -2^5 \sum_{n_{odd}}^{\infty} \frac{L}{8\pi^2\alpha'} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \left(\frac{1}{4\pi^2\alpha'\tau_2}\right)^2 e^{-\frac{n^2\beta^2}{4\pi\alpha'\tau_2}} \prod_{m=1}^{\infty} \left|\frac{1+e^{2\pi i\tau m}}{1-e^{2\pi i\tau m}}\right|^8 \times$$
$$\times \prod_{m=-\infty}^{\infty} \left\{ \frac{1+\exp\left[-2\pi\tau_2\sqrt{\left(\frac{n f\beta\sin\alpha}{2\sqrt{2}\pi\tau_2}\right)^2 + \left(m+\frac{n\beta f\cos\alpha}{2\sqrt{2}\pi\tau_2}\right)^2} + 2\pi i\tau_1 m\right]}{1-\exp\left[-2\pi\tau_2\sqrt{\left(\frac{n f\beta\sin\alpha}{2\sqrt{2}\pi\tau_2}\right)^2 + \left(m+\frac{n\beta f\cos\alpha}{2\sqrt{2}\pi\tau_2}\right)^2} + 2\pi i\tau_1 m\right]} \right\}^4 . \tag{5.5}$$

Here, because of the fermion zero modes, the ground state is degenerate and the free energy can be computed using $F_{\rm susy}$ defined in equation (2.3).

The modular properties of the partition function in (5.5) can be derived as in section 3. Consider

$$Z(\tau_1,\tau_2,x) = \prod_{m=-\infty}^{\infty} \left(\frac{1+e^{-2\pi\tau_2\sqrt{x^2+(m+b)^2}+2\pi i\tau_1(m+b)+2\pi i a}}{1-e^{-2\pi\tau_2\sqrt{x^2+(m+b)^2}+2\pi i\tau_1(m+b)+2\pi i a}}\right) |\theta_4(0,2\tau)|^{-2} . \tag{5.6}$$

In our case $a=0$, $b=\frac{n\beta f\cos\alpha}{2\sqrt{2}\pi\tau_2}$ and $x=\frac{n\beta f\sin\alpha}{2\sqrt{2}\pi\tau_2}$. (5.6) can be rewritten in terms of the definition (3.1) as

$$Z(\tau_1,\tau_2,x) = \frac{Z_{\frac{1}{2},b}(\tau_1,\tau_2,x)}{Z_{0,b}(\tau_1,\tau_2,x)} |\theta_4(0,2\tau)|^{-2} \tag{5.7}$$

From the modular property of $Z_{a,b}(\tau_1,\tau_2,x)$, eq. (3.5), it follows that

$$\begin{aligned}\ln Z\left(\tau_1,\tau_2,\frac{n\beta f\sin\alpha}{\sqrt{2}2\pi\tau_2}\right) &= \ln Z_{-b,\frac{1}{2}}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{n\beta f\sin\alpha|\tau|}{2\pi\sqrt{2}\tau_2}\right) - \\ &- \ln Z_{-b,0}\left(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2},\frac{n\beta\sin\alpha f|\tau|}{2\pi\sqrt{2}\tau_2}\right) + \\ &+2\pi\frac{\tau_2}{|\tau|^2}\left[\Delta_{\frac{1}{2}}\left(\frac{n\beta f\sin\alpha|\tau|}{2\pi\sqrt{2}\tau_2}\right) - \Delta_0\left(\frac{n\beta f\sin\alpha|\tau|}{2\pi\sqrt{2}\tau_2}\right)\right] - \\ &-2\ln\left|\theta_2(0,-\frac{1}{2\tau})\right| + \ln 2|\tau|\end{aligned} \tag{5.8}$$

The first two terms in (5.8) behave smoothly in the $\tau_1 \to 0$, $\tau_2 \to 0$ limit. Moreover

$$\left|\theta_2(0,-\frac{1}{2\tau})\right| \to \exp\left(-\frac{\pi\tau_2}{4|\tau|^2}\right) .$$

Consequently, taking into account the definition of the Casimir energies, for the Hagedorn temperature we get

$$\frac{\beta_H^2}{4\pi\alpha'} = \frac{4\beta_H f\sin\alpha}{\sqrt{2}\pi} \sum_{p=1}^{\infty} \frac{[1-(-1)^p]}{p} K_1\left(\frac{p\beta_H f\sin\alpha}{\sqrt{2}}\right) + \pi . \tag{5.9}$$

It is interesting to note that this equation depends on the angle α only through $f\sin\alpha$, the RR field strenght.[3]

[3]We thank Y. Sugawara for pointing out the misprint present in the previous version of the Paper.

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Keeping only the two leading terms in the expansion for small f of (5.9) we get the Hagedorn temperature

$$T_H = \frac{1}{2\pi\sqrt{2\alpha'}}\left(1+\sqrt{\alpha'}f\sin\alpha+(1-2\ln 2)\alpha' f^2\sin^2\alpha\right). \tag{5.10}$$

In the case of purely NS background, corresponding to $\alpha = 0$, we recover the well known superstring Hagedorn temperature for the flat background.

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A. Alternative derivation of eq. (4.14)

The same expression for the difference $S_b(x) - S_f(x)$ can be obtained using a completely different procedure. The series $S_b(x) - S_f(x)$ can in fact be obtained also from the formula

$$S_b(x) - S_f(x) = -\frac{d}{dx}\left(x^2\int_0^{\pi/x} dt' \int_0^{t'} dt \sum_{p=1}^{\infty} K_0(xp)\cos pxt\right). \tag{A.1}$$

Using the fact that [61]

$$\begin{aligned}\sum_{p=1}^{\infty} K_0(xp)\cos pxt = {}& \frac{1}{2}\left(\gamma+\ln\frac{x}{4\pi}\right)+\frac{\pi}{2x\sqrt{1+t^2}}+\\ &+\frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2+(2l\pi-tx)^2}}-\frac{1}{2l\pi}\right\}+\\ &+\frac{\pi}{2}\sum_{l=1}^{\infty}\left\{\frac{1}{\sqrt{x^2+(2l\pi+tx)^2}}-\frac{1}{2l\pi}\right\}\end{aligned} \tag{A.2}$$

equation (A.1) becomes

$$\begin{aligned}S_b(x) - S_f(x) = {}& -\frac{\pi^2}{4x}-\frac{\pi}{2}+\frac{\pi}{2}\left(\frac{\sqrt{x^2+\pi^2}}{x}+\frac{x}{\pi+\sqrt{x^2+\pi^2}}\right)-\\ &-\pi x\sum_{l=1}^{\infty}\left(\frac{1}{2l\pi+\sqrt{x^2+(2l\pi)^2}}-\frac{1}{(2l+1)\pi+\sqrt{x^2+(2l+1)^2\pi^2}}\right)\\ ={}& \frac{\pi^2}{4x}+\frac{\pi}{2}-\pi x\sum_{l=0}^{\infty}\left(\frac{1}{2l\pi+\sqrt{x^2+(2l\pi)^2}}-\frac{1}{(2l+1)\pi+\sqrt{x^2+(2l+1)^2\pi^2}}\right)\\ ={}& \frac{\pi^2}{4x}-\frac{\pi}{2}-\pi x\sum_{k=1}^{\infty}\frac{(-1)^k}{\pi k+\sqrt{x^2+\pi^2k^2}}.\end{aligned} \tag{A.3}$$

Expanding for small values of x, it is easy to prove that equation (A.3) becomes precisely equation (4.14)

$$S_b(x) - S_f(x) = \frac{\pi^2}{4x} - \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\sqrt{\pi}} \Gamma\left(k - \frac{1}{2}\right) \left(\frac{x}{\pi}\right)^{2k-1} T_{2k-1} \tag{A.4}$$

where $T_s = \left(2^{1-s} - 1\right) \zeta(s)$ and $T_1 = -\ln 2$ [63].

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